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A NOTE ON SOME INTEGRALS INVOLVING HERMITE POLYNOMIALS AND THEIR APPLICATIONS

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Abstract. Closed-form expressions of some integrals involving Hermite polynomials that are encountered in many problems in physics field are obtained. The derived formulae are expressed in terms of Hermite polynomials and can be used as alternative expressions instead of the infinite series representation of the integrals. As applications some well-known integrals which correspond to some catastrophe caustic optics such as Olver, Pearcey, Swallowtail and Butterfly beams are treated. New amplitude representations of the mentioned fields are derived.

Keywords: Hermite polynomials, Olver beam, Pearcey beam, Swallowtail beam, Butterfly beam, Catastrophe optics.

AMS Subject Classification: 33B15, 33C10, 33C15.

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1 Introduction

In past years, many authors devoted to optics domain have gained interest in introducing new classes of paraxial optical beams whose profiles allow wide applications in physics (see Belafhal et al. (2015), Berry et al. (1979), Cai et al. (2003), Durnin et al. (1987), Karimi et al. (2008), Ring et al. (2012), Siviliglou et al. (2007) and Zannoti et al. (2017)). Recently, Habibi et al. (2018) introduced a new paraxial mode named as the Mainardi beam. The authors derived the expression of the amplitude field for describing the propagating Mainardi beam through the free-space and fractional fourier transform system. The formulae that they obtained contains infinite series expressions in the integrand functions (see Habibi et al. (2018)). These latter have been used with numerical method to illustrate the characteristics of the beam versus its pertinent parameters. However, as it is well-known, the result would be more pertinent if such field characteristics are expressed in closed-form, i.e, in terms of well-known special functions. The mentioned equations in Habibi et al. (2018) can be, for instance, rewritten in terms of Hermite polynomials if one uses new methods to evaluate the integrals, as it will be demonstrated in the following. The procedure may permit us to avoid as possible the use of the infinite expansion series representations. It is worthy noting that a similar work concerning some integrals used in laser physics, involving the product of Bessel functions has been recently published (Belafhal & Hennani, 2011).

The present paper is aimed to evaluate some diffraction integrals that are connected to Hermite polynomials, and which are recurrent in the evaluation of the characteristics for a propagating beam. We evaluate some integrals in terms of Hermite functions. Integral representations in connection with caustic optics are considered as particular cases of the evaluated integrals, and their corresponding results are presented. A brief conclusion is given in the end of the paper.

In caustic optics, we need to evaluate some integrals involving a product of Hermite polynomials and Gaussian weight. In what follows, we will be interested in evaluating the integral formulae

$$I_1 = \int_{-\infty}^{\infty} x^l e^{-px^2+2qx} dx, \tag{1}$$

$$I_2 = \int_{-\infty}^{\infty} H_m(\alpha x) x^l e^{-px^2+2qx} dx, \tag{2}$$

and

$$I_3 = \int_{-\infty}^{\infty} H_m(\alpha x) H_n(\alpha x) x^l e^{-px^2+2qx} dx, \tag{3}$$

where $p > 0$ and α; q are complex numbers.

The Hermite polynomials are defined by Gradshteyn & Ryzhik (1994)

$$H_l(z) = l! \sum_{k=0}^{[l/2]} \frac{(-1)^k}{k!(l-2k)!} (2z)^{l-2k}, \tag{4}$$

where $[l/2]$ is the truncated part of $l/2$.

2 Main results

In this section, we establish three integral formulae I_1, I_2 and I_3 involving Hermite polynomials which are recurrent in the evaluation of the characteristics for a propagating beam.

2.1 Evaluation of the integral I_1

Theorem 1. *Let $p > 0$. Then we have*

$$I_1 = \int_{-\infty}^{\infty} x^l e^{-px^2+2qx} dx = e^{\frac{q^2}{p}} \sqrt{\frac{\pi}{p}} \left(\frac{1}{2i\sqrt{p}}\right)^l H_l\left(\frac{iq}{\sqrt{p}}\right). \tag{5}$$

Proof. This integral is given in many text books, to the best of our knowledge, as series expansion by (see Gradshteyn & Ryzhik (1994))

$$\int_{-\infty}^{\infty} x^l e^{-px^2+2qx} dx = l! e^{\frac{q^2}{p}} \sqrt{\frac{\pi}{p}} \left(\frac{q}{p}\right)^l \sum_{k=0}^{[l/2]} \frac{(-1)^k}{k!(l-2k)!} \left(\frac{p}{4q^2}\right)^k, \tag{6}$$

where $p > 0$.

Recalling the expansion formula of the Hermite polynomial of order n given by (4) and putting the substitution $z = \frac{iq}{\sqrt{p}}$, one can rewrite the expression H_l as

$$H_l\left(\frac{iq}{\sqrt{p}}\right) = l! \left(\frac{2iq}{\sqrt{p}}\right)^l \sum_{k=0}^{[l/2]} \frac{(-1)^k}{k!(l-2k)!} \left(\frac{2iq}{\sqrt{p}}\right)^{-2k}. \tag{7}$$

From this last equation, one can deduce the following relation:

$$\sum_{k=0}^{[l/2]} \frac{(-1)^k}{k!(l-2k)!} \left(\frac{p}{4q^2}\right)^k = \frac{1}{l!} \left(\frac{\sqrt{p}}{2iq}\right)^l H_l\left(\frac{iq}{\sqrt{p}}\right). \tag{8}$$

Now by substituting (8) into (6), one obtains the value of I_1 in terms of Hermite polynomial. □

2.2 Evaluation of the integral I_2

Lemma 1. *The following holds for* $\mid \alpha \mid < 1$*:*

$$\sum_{p=0}^{[m/2]} \frac{(-1)^p m!}{p!(m-2p)!}(-i\alpha)^{m-2p} H_{m-2p}(iy) = \left(1-\alpha^2\right)^{\frac{m}{2}} H_m\left(\frac{\alpha y}{\sqrt{1-\alpha^2}}\right). \tag{9}$$

Proof. Firstly, we consider the following integral:

$$I = e^{-y^2}\int_{-\infty}^{\infty} H_m(\alpha x)e^{-x^2+2yx}dx. \tag{10}$$

With the help of (4), this integral becomes

$$I = e^{-y^2}\sum_{p=0}^{[m/2]} \frac{(-1)^p m!}{p!(m-2p)!}(2\alpha)^{m-2p}\int_{-\infty}^{\infty} x^{m-2p}e^{-x^2+2yx}dx. \tag{11}$$

By taking $p = 1, q = y$ and $n = m-2p$ in (5), and using 7.374 (8) (page 797 of Gradshteyn & Ryzhik (1994)), we get

$$\int_{-\infty}^{\infty} H_m(\alpha x)e^{-(x-y)^2}dx = \sqrt{\pi}\left(1-\alpha^2\right)^{\frac{m}{2}} H_m\left(\frac{\alpha y}{\sqrt{1-\alpha^2}}\right), \tag{12}$$

from which, it is easy to find (9). □

Corollary 1. *By substituting in* (9), $\alpha = \frac{ia}{\sqrt{1-a^2}}$*, it is easy to prove the corresponding result of Bailey (1948)*

$$H_m(iax) = m!\sum_{p=0}^{[m/2]} \frac{(-1)^p}{p!(m-2p)!}(ia)^{m-2p}(1+a^2)^p H_{m-2p}(x). \tag{13}$$

Theorem 2. *For* $p > 0$*, the following transformation holds:*

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} x^l H_m(\alpha x)e^{-px^2+2qx}dx \\ &= \frac{e^{\frac{q^2}{p}}}{2^l}\sqrt{\frac{\pi}{q}}\sum_{k=0}^{[m/2]} \frac{(-1)^k m!}{k!(m-2k)!}\left(\frac{\alpha}{i\sqrt{p}}\right)^{m+l-2k} H_{m+l-2k}\left(\frac{iq}{\sqrt{p}}\right), \end{aligned} \tag{14}$$

Proof. By using the expansion formula in (4) and in view of the expression of I_2, we obtain

$$I_2 = \sum_{k=0}^{[m/2]} m!\frac{(-1)^k}{k!(m-2k)!}(2\alpha)^{m-2k} I_{mk}, \tag{15}$$

where

$$I_{mk} = \int_{-\infty}^{\infty} x^{m-2k+l}e^{-px^2+2qx}dx. \tag{16}$$

Now, with the help of Theorem 1, we deduce the expression

$$I_{mk} = e^{\frac{q^2}{p}}\sqrt{\frac{\pi}{p}}\left(\frac{1}{2i\sqrt{p}}\right)^{m-2k+l} H_{m+l-2k}\left(\frac{iq}{\sqrt{p}}\right). \tag{17}$$

Finally, by using the Lemma 1 and (17), we obtain (14). This completes the proof. □

Corollary 2. *For* $l = 0$ *and with the help of Lemma 1,* (14) *becomes*

$$\int_{-\infty}^{\infty} H_m(\alpha x)e^{-px^2+2qx}dx = e^{\frac{q^2}{p}}\sqrt{\frac{\pi}{p}}\left(1-\frac{\alpha^2}{p}\right)^{\frac{m}{2}} H_m\left(\frac{\alpha q}{p\sqrt{1-\frac{\alpha^2}{p}}}\right). \tag{18}$$

2.3 Evaluation of the integral I_3

Theorem 3. *For $p > 0$, the following transformation holds:*

$$
\begin{aligned}
I_3 =& \int_{-\infty}^{\infty} H_m(\alpha x) H_n(\alpha x) x^l e^{-px^2+2qx} dx \\
=& e^{\frac{q^2}{p}} \sqrt{\frac{\pi}{p}} \frac{\alpha^n}{2^l} \times \sum_{k=0}^{[n/2]} \sum_{k'=0}^{[m/2]} \frac{(-1)^{k+k'} n! m!}{k! k'! (n-2k)! (m-2k')!} \frac{H_{m+n+l-2k-2k'}\left(\frac{iq}{\sqrt{p}}\right)}{\alpha^{2k} (i\sqrt{p})^{m+n+l-2k-2k'}}.
\end{aligned} \tag{19}
$$

Proof. The use of the expansion in (4) yields

$$I_3 = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!(n-2k)!} (2\alpha)^{n-2k} I_{mkl}, \tag{20}$$

where

$$I_{mkk} = \int_{-\infty}^{\infty} H_m(\alpha x) x^{l+n-2k} e^{-px^2+2qx} dx. \tag{21}$$

By applying Theorem 2, (20) can be rewritten in the form of (19). This completes the proof. □

3 Applications to catastrophe optics

3.1 Generalities

Generally speaking, to describe the propagation characteristics of an optical beam propagating through a paraxial ABCD system, one may use intuitively the well-known Huygens—Fresnel diffraction integral. During the evaluation of the integral, one might need to evaluate expressions that are proportional to I_1, I_2, or I_3. For example, in Habibi et al. (2018), the authors have obtained (19) and (22), whose integrand functions can be evaluated, according our method, in terms of Hermite polynomials.

In catastrophe optics theory Hobbs et al. (1987), some caustic patterns such as Pearcey, Swollowtail, Butterfly and Olver beams possess amplitude fields that are defined by the following integral representation

$$U_n(t, s, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{a(i\lambda)^n t - \frac{1}{2} s t \lambda^2 + i\lambda x} d\lambda. \tag{22}$$

Taking into account the result of the above section, one can express this last integral in terms of Hermite polynomials. In fact, in a first step, the integral expression can be rewritten as

$$U_n(t, s, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-p\lambda^2 + 2q\lambda + a(i\lambda)^n t} d\lambda, \tag{23}$$

with $p = \frac{1}{2} s.t$ and $q = \frac{ix}{2}$.

Recalling the expansion formula of the exponential function

$$e^{a(i\lambda)^n t} = \sum_{-\infty}^{\infty} \frac{[a(i\lambda)^n t]^j}{j!}, \tag{24}$$

and substituting this last expression into (23), yields

$$U_n(t, s, x) = \frac{1}{2\pi} \sum_{j=0}^{\infty} \frac{[a i^n t]^j}{j!} I_j, \tag{25}$$

where

$$I_j = \int_{-\infty}^{\infty} e^{-p\lambda^2+2q\lambda}.(\lambda)^{nj} d\lambda. \tag{26}$$

Taking (5) into account, the integral in (26) reads

$$I = e^{-\frac{x^2}{2st}} \sqrt{\frac{2\pi}{st}} \left(\frac{1}{j\sqrt{2st}}\right)^{n.j} H_{n,j}\left(-\frac{x}{\sqrt{2st}}\right). \tag{27}$$

On substituting (27) into (25), we derive

$$\begin{aligned} U_n(t,s,x) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-p\lambda^2+2q\lambda+a(i\lambda)^n t} d\lambda \\ =& \frac{1}{\sqrt{2\pi st}} e^{-\frac{x^2}{2st}} \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{at}{(\sqrt{2st})^n}\right)^j H_{n,j}\left(-\frac{x}{\sqrt{2st}}\right). \end{aligned} \tag{28}$$

This last result is valuable because it will permit to evaluate the amplitude field expression of some catastrophe beams in an alternative way with Hermite polynomials.

From (28), one can prove that the generating functions of Hermite polynomials Hernandez-Del-Valle (2010) of the form $\sum_{j=0}^{\infty} \frac{z^j}{j!} H_{j,n}(-\frac{x}{\sqrt{2st}})$ are equivalent to Airy-heat functions, defined as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{a\lambda^n - \frac{\lambda^2 t}{2} + i\lambda x} d\lambda,$$

that is

$$\sum_{j=0}^{\infty} \frac{z^j}{j!} H_{j,n}\left(-\frac{x}{\sqrt{2st}}\right) = \sqrt{2\pi st}\, e^{-\frac{x^2}{2st}} \int_{-\infty}^{\infty} e^{at(i\lambda)^n - \frac{st}{2}\lambda^2 + i\lambda x} d\lambda,$$

and we can deduce the following expression for $n = 3$

$$e^{\left(\frac{s^3 t}{12}+\frac{sx}{2}\right)t^{-\frac{1}{3}}} A_i\left\{t^{-\frac{1}{3}}\left(x+\frac{s^2 t}{4}\right)\right\} = \frac{1}{4\sqrt{\pi}(st)^2} \sum_{j=0}^{\infty} \frac{(-\frac{t}{3})^j}{j!} H_{3,j}\left(-\frac{x}{\sqrt{2st}}\right). \tag{29}$$

3.2 Alternative amplitude expression for some catastrophe beams

(a) The integral representation of the Olver beams family Belafhal et al. (2015) is defined as

$$U_{Ol}(s,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{a(i\lambda)^{m+3} - \frac{1}{2}s\lambda^2 + 2i\lambda x} d\lambda. \tag{30}$$

One can note that this last equation can be regarded as a particular case of (22) with $m+3$, $a = 1$ and $t = 1$, so its value is straightforwardly obtained from (28) as

$$U_{Ol}(s,x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{1}{(\sqrt{2s})^{m+3}}\right)^j H_{(m+3),j}\left(-\frac{x}{\sqrt{2s}}\right). \tag{31}$$

(b) Substituting $n = 4$, $a = 1$, $t = i$ and $s = -2.\frac{y}{y_0}$, $\nu = \frac{x}{x_0}$ in the expression of (22) will give the integral representation of the Pearcey beam Ring et al. (2012),

$$P\left(\frac{x}{x_0}, \frac{y}{y_0}\right) = \int_{-\infty}^{\infty} e^{i\left(\lambda^4 + \frac{y}{y_0}\lambda^2 + \frac{x}{x_0}\lambda\right)} d\lambda. \tag{32}$$

Therefore, from the result of (28) this last equation can be expressed as

$$\begin{aligned} P\left(\frac{x}{x_0},\frac{y}{y_0}\right) =& U_4(1,s,\nu) \\ =& \sqrt{\frac{iy_0}{4\pi y}} e^{-i\frac{(\frac{x}{x_0})^2}{4\frac{y}{y_0}}} \sum_{j=0}^{\infty}\frac{1}{j!}\left[-\frac{i}{16}\left(\frac{y_0}{y}\right)^2\right]^j H_{4,j}\left(-\frac{\frac{x}{x_0}}{\sqrt{-2i\frac{y}{y_0}}}\right). \end{aligned} \tag{33}$$

(c) The case of $n=5$, $a=1$, $t=1$, $s=-2i.\frac{y}{y_0}$ and $\nu=\frac{x}{x_0}$ gives the well-known amplitude expression of the Swallowtail beam (see Ring et al. (2012), Zannoti et al. (2017)),

$$S_W\left(\frac{x}{x_0},\frac{y}{y_0}\right) = \int_{-\infty}^{\infty} e^{i\left(\lambda^5+\frac{y}{y_0}\lambda^2+\frac{x}{x_0}\lambda\right)}d\lambda. \tag{34}$$

Therefore, one can write

$$\begin{aligned} S_W\left(\frac{x}{x_0},\frac{y}{y_0}\right) =& 2\pi\, U_5\left(1,-2i\frac{y}{y_0},\frac{x}{x_0}\right) \\ =& \sqrt{\frac{iy_0}{4\pi y}} e^{-i\frac{\left(\frac{x}{x_0}\right)^2}{4\frac{y}{y_0}}} \sum_{j=0}^{\infty}\frac{1}{j!}\left[-\frac{i}{16}\left(\frac{y_0}{y}\right)^2\right]^j H_{5,j}\left(-\frac{\frac{x}{x_0}}{\sqrt{-2i\frac{y}{y_0}}}\right). \end{aligned} \tag{35}$$

(d) A Butterfly beam is defined by the amplitude expression Ring et al. (2012)

$$B_u\left(\frac{x}{x_0},\frac{y}{y_0}\right) = \int_{-\infty}^{\infty} e^{i\left(\lambda^6+\frac{y}{y_0}\lambda^2+\frac{x}{x_0}\lambda\right)}d\lambda. \tag{36}$$

Corresponding to the values $n=6$, $a=1$, $t=-i$, $s=-2.\frac{y}{y_0}$, $\nu=\frac{x}{x_0}$ into (22), one obtains in this case

$$\begin{aligned} B_u\left(\frac{x}{x_0},\frac{y}{y_0}\right) =& 2\pi U_6\left(-i,2\frac{y}{y_0},\frac{x}{x_0}\right) \\ =& \sqrt{\frac{iy_0}{4\pi y}} e^{-i\frac{(\frac{x}{x_0})^2}{4\frac{y}{y_0}}} \sum_{j=0}^{\infty}\frac{1}{j!}\left[-\frac{i}{16}\left(\frac{y_0}{y}\right)^2\right]^j H_{6,j}\left(-\frac{\frac{x}{x_0}}{\sqrt{-2i\frac{y}{y_0}}}\right). \end{aligned} \tag{37}$$

4 Conclusion

We have evaluated analytically, some interesting integral expressions that are recurrent in problems dealing with Huygens-Fresnel diffraction. The integrals are expressed in terms of Hermite polynomials. Applications to catastrophe optics theory allowed us to obtain alternative formulations of the amplitude for some caustics beams. The obtained formulas are believed to be new and useful for the laser specialists.

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